Pergamon

0021–8928(94)00123-5

J. Appl. Maths Mechs, Vol. 58, No. 6, pp. 945–954, 1994

0021–8928/94 $24.00+0.00

THE SOLVABILITY OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS WITH SLIDING FRICTION†

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(*Received* 1 *December* 1993)

The problem of the existence of local classical solutions of the equations of motion of holonomic systems with sliding friction is investigated. The conditions for the equations of motion to be solvable for the accelerations enable estimates to be obtained of the coefficients of friction, within the framework of which (from the point of view of the existence of solutions) Coulomb's laws can be used to describe the dynamics of mechanical systems with sliding friction.

Under certain conditions the direct use of Coulomb's law of dry friction involves introducing friction forces which depend on the normal reactions, which are functions of the accelerations. It is not always possible to solve these equations of motion for the accelerations, and the solution is not always unique. Hence, for the equations of motion this does not lead to supplementing the right-hand sides at certain points of the discontinuity or leads to their non-unique supplementation even for autonomous mechanical systems with holonomic stationary couplings.

The first phenomenon of this kind in the history of mechanics was discovered by Painlevé in his lectures on friction [1] and was a paradox which gave rise to discussions as well as theoretical and experimental investigations. Problems of the dynamics of systems with dry friction can now be solved in regions where the right-hand sides of the equations of motion can be defined by Coulomb's laws.

The general theory of the motion of mechanical systems with friction was set up by [1] and was developed in a number of well-known papers ([2–7], etc.), in which the Euler–Lagrange principle of possible displacements, Lagrange's method, and Gauss's principle of at least constraint were extended to systems with friction.

1. THE EQUATIONS OF MOTION

Suppose we are given a mechanical system with k degrees of freedom, constrained by holonomic (generally speaking, retaining and time-dependent) ideal couplings with forces of sliding friction, added to active forces. The equations of its motion can be written in Lagrangian form

$$\frac{d}{dt}\frac{\partial T_a}{\partial \dot{q}^i} - \frac{\partial T_a}{\partial q^i} = Q_i, \quad i = 1,\ldots,k \tag{1.1}$$

Here $\dot{q}^i(t) = dq^i/dt$ are generalized velocities and T_a is the kinetic energy of the system in motion with respect to an internal system of coordinates, which is the sum $T_a = T + T_1 + T_0$ of a positive definite quadratic form T of the generalized velocities in a certain region of variation of the variables (t, q)

$$T = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}(t,q)\dot{q}^i\dot{q}^j, \quad (a_{ij}(t,q) = a_{ji}(t,q)) \tag{1.2}$$

of the linear form of the generalized velocities

†*Prikl. Mat. Mekh.* Vol. 58, No. 6, pp. 3–13, 1994.

$$T_1 = \sum_{i=1}^{k} a_i(t,q)\dot{q}^i$$

and of the function $T_0 = T_0(t, q)$. The functions $a_{ij}(t, q)$, $a_i(t, q)$, $T_0(t, q)$ are assumed to be continuously differentiable with respect to the set of arguments in the region of definition of the space R^{k+1}.

The forces F_μ acting at a point of the system with radius vectors $r_\mu(t, q)$ include the active forces F_μ^A and the reaction forces of the couplings, which consist of the normal-reaction forces F_μ^N and the forces of sliding friction F_μ^T: $F_\mu = F_\mu^A + F_\mu^N + F_\mu^T$. The generalized forces are calculated as the coefficients of δq^i in the expression for the virtual work of the forces F_μ, acting at a point of the system. They are linear functions in F_μ^A and F_μ^T with coefficients that depend continuously on t and q

$$Q_i = \sum_\mu \left(\frac{\partial r_\mu}{\partial q^i} F_\mu^A \right) + \sum_\mu \left(\frac{\partial r_\mu}{\partial q^i} F_\mu^T \right) = Q_i^A + Q_i^T$$

We will introduce the following notation: $q = (q^1, \dots, q^k)^T$, $\ddot{q} = (\ddot{q}^1, \dots, \dot{q}, \ddot{q}^k)^T$, $Q = (Q_1, \dots, Q_k)^T$ are the vectors of the generalized coordinates, velocities, accelerations and forces, and $g = (g_1, \dots, g_k)^T$ is a continuous vector function

$$g_i(t,q,\dot{q}) = \sum_{j=1}^{k} \left(\frac{\partial a_j}{\partial q^i} - \frac{\partial a_i}{\partial q^j} \right) \dot{q}^j + \frac{1}{2} \sum_{\nu=1}^{k} \sum_{j=1}^{k} \frac{\partial a_{\nu j}}{\partial q^i} \dot{q}^\nu \dot{q}^j -$$

$$- \sum_{j=1}^{k} \left(\frac{\partial a_{ij}}{\partial t} + \sum_{\nu=1}^{k} \frac{\partial a_{ij}}{\partial q^\nu} \dot{q}^\nu \right) \dot{q}^j + \frac{\partial T_0}{\partial q^i} - \frac{\partial a_i}{\partial t}$$

describing generalized gyroscopic forces, the transferred forces of inertia and certain other terms. Taking into account the possible dependence of the generalized forces of friction Q_i^T on the generalized normal reactions of the couplings N_i, and the dependence of the latter on the generalized accelerations, and assuming, as is usually done, that the generalized active forces Q_i^A depend continuously only on the generalized coordinates, velocities and time, we can write

$$Q = Q^A(t,q,\dot{q}) + Q^T(t,q,\dot{q},\ddot{q})$$

System (1.1) can then be written in the form of the following equation

$$A(t,q)\ddot{q} = g(t,q,\dot{q}) + Q^A(t,q,\dot{q}) + Q^T(t,q,\dot{q},\ddot{q}), \quad (t,q,\dot{q}) \in \Omega \tag{1.3}$$

Here $A(t, q)$ is the matrix of the coefficients of the quadratic form (1.2) (the coefficients of inertia) with determinant $\det A(t, q) > 0$ for any t, q in a certain region of R^{k+1}, $Q^A(t, q, \dot{q})$ include the potential forces, the forces of radial correction, and the forces of resistance of the dampers and the medium (we will not give a detailed description of these forces), and $Q_i^T(t, q, \dot{q}, \ddot{q})$ are the generalized forces of sliding friction.

2. GENERALIZED FORCES OF SLIDING FRICTION

Suppose the couplings and generalized coordinates $q^1, \dots, q^k$ are such (they can be so chosen), that only a change in q defines the sliding when a force of friction occurs and each of the generalized forces of friction Q_i^T depends explicitly on only the corresponding (one) generalized velocity $\dot{q}^i$ and the normal reaction N_i (possibly depending also on other generalized velocities and accelerations). By Coulomb's laws (see [2, 3]), in the case of motion with generalized velocities $\dot{q}^s(t) \neq 0$, $(s = 1, \dots, k_*$, $1 \geqslant k_* \geqslant k)$, the generalized forces of sliding friction Q_s^T are expressed by the following formulae

$$Q_s^{T1} = -f_s |N_s| \operatorname{sgn} \dot{q}^s, \quad (s = 1, \dots, k_*)$$

in terms of the coefficients of friction (during motion) f_s and the moduli of the normal reactions $|N_s|$

at points where the rubbing bodies come in contact. The latter are found [5] from Lagrange's equations with Lagrange multipliers λ_j, ($j = 1, \ldots, k^*, k^* \geqslant k_*$) for a system with kinetic energy T_a and additional generalized coordinates $q_* = (q_*^1, \ldots, q_*^{k^*})$, obtained from the initial coordinates after an imaginary freeing of the couplings $q_*^1 = 0, \ldots, q_*^{k^*} = 0$ ($q_* = 0, \ddot{q}_* = 0$) which give rise to the required reactions. Then we have

$$\lambda_i = \left[\frac{d}{dt}\frac{\partial T_{a*}}{\partial \dot{q}_*^j} - \frac{\partial T_{a*}}{\partial q_*^j} - Q_j^*(t, q, q_*, \dot{q}, \dot{q}_*)\right]_{q_*=0, \dot{q}_*=0, \ddot{q}_*=0} =$$

$$= \sum_{i=1}^{k} a_{k+ji}^*(t, q, 0)\ddot{q}^i - g_j^*(t, q, \dot{q}) - Q_j^*(t, q, \dot{q}), \quad (j = 1, \ldots, k^*)$$

The moduli of the normal reactions are determined by the equations $|N_s| = |\lambda_s|$ for sliding with velocity $\dot{q}^s$ over the surface $q_*^s = 0$, $s = 1, \ldots, k'$, $0 \leqslant k' \leqslant k_*$; $|N_s| = \sqrt{(\lambda_j^2 + \lambda_{j'}^2)}$, if N_s is obtained from the sliding of mutually orthogonal reactions of the smooth couplings $q_*^j = 0, q_*^{j'} = 0$ with moduli $|\lambda_j|$, $|\lambda_{j'}|$, $s = k' + 1, \ldots, k_*$, $i, j = k' + 1, \ldots, k^*$ when the body rotates around the $q^j = 0, q^{j'} = 0$ axis or when the point slides over a spatial curve.

Note that the functions $|N_s|$ are continuous with respect to $\ddot{q}$ when $|N_s| \neq 0$ they are continuously differentiable with respect to $\ddot{q}$.

Thus, we have the following equation for the generalized forces of sliding friction during motion

$$Q_s^{T1} = -f_s(t, q^s, \dot{q}^s)|N_s(t, q, \dot{q}, \ddot{q})|\operatorname{sgn}\dot{q}^s \text{ when } \dot{q}^s \neq 0, s = 1, \ldots, k_* \tag{2.1}$$

The coefficient of friction during motion $f_s(t, q^s, \dot{q}^s) > 0$ (for dry friction it is usually taken to be equal to a positive constant, but if there is a lubricant or some other liquid on the surfaces in contact and also when there is a change in the temperature of the surfaces or the processing on different parts of the surface are of different cleanliness, etc., it may be a function of the variables t, q^s and $\dot{q}^s$).

For the remaining $k_* < s \leqslant k$ we will assume that there is no friction

$$f_s = 0, \quad Q_s^T = 0 \quad s = k_* + 1, \ldots, k$$

The equations of motion (1.3) and (2.1) in the region $\dot{q}^s \neq 0$, $s = 1, \ldots, k_*$ can be reduced to the form

$$\ddot{q} = A^{-1}(t, q)R(t, q, \dot{q}, \ddot{q}, f)$$

where R is the right-hand side of (1.3) with force of friction (2.1). When $f \triangleq (f_1, \ldots, f_{k_*}) = 0$ the latter are uniquely solvable for $\ddot{q}$. Taking into account the form of the functions $|N_s|$ and the structure of the right-hand side of the equations, we assert that for sufficiently small $f_1, \ldots, f_{k_*}$ the mapping $\ddot{q} \to A^{-1}(t, q)R(t, q, \dot{q}, \ddot{q}, f)$ is a compressive mapping. Simple calculations using the principle of compressive mappings [8] show that the equations are uniquely solvable for $\ddot{q}$ in a certain neighbourhood of each point $(t, q, \dot{q})$ for sufficiently small $f_1, \ldots, f_{k_*}$ and can be reduced to the form

$$\ddot{q} = G(t, q, \dot{q}, f) \tag{2.2}$$

where G is a function that is continuous over the set of its arguments.

By Peano's theorem for any initial data from the region considered there is a locally classical solution of the equations obtained. Hence, in the region $\dot{q}^s \neq 0$, $s = 1, \ldots, k_*$, $\det A(t, q) > 0$ the generalized forces of friction can be supplemented by functions which are independent of the accelerations $\ddot{q}^s$ (if the coefficients of friction are sufficiently small)

$$Q_s^T(t, q, \dot{q}) = -f_s(t, q^s, \dot{q}^s)|N_s(t, q, \dot{q}, G(t, q, \dot{q}, f))|\operatorname{sgn}\dot{q}^s$$

When $|N_s| \neq 0$, and $|N_s(t, \cdot)|$ and the other functions on the right-hand sides of the equations of motion are continuously differentiable with respect to q, $\dot{q}$, $\ddot{q}$, the function G is continuously differentiable with respect to the same variables (according to the theorem on implicit functions), and the solutions of Eq. (2.2) are unique and define the motions of mechanical system (1.1) [9].

Suppose now that the sliding velocity of the rubbing body at a certain instant of time is zero. By the rules of classical mechanics [3, p. 107] we will assume that the coefficients of static friction f_s^0 are equal to the coefficients of friction in motion, i.e.

$$f_s^0(t,q^s)=f_s(t,q^s,0),\quad s=1,\dots,k_*$$

If $\dot{q}^s(t) = 0$ for a certain value of the superscript $1 \leqslant s \leqslant k^*$, we will assume $\ddot{q}^s = 0$ and we will calculate the generalized force of sliding friction using the coefficients of static friction

$$Q_s^{T0}(t,q,\dot{q},\ddot{q})\overset{\Delta}{=}\sum_{j=1,j\neq s}^{k} a_{sj}(t,q)\ddot{q}^j-\left[g_s(t,q,\dot{q})-Q_s^A(t,q,\dot{q})\right]_{\dot{q}^s=0}$$

If the following inequality is satisfied

$$\left|Q_s^{T0}(t,q,\dot{q},\ddot{q})\right|\leq f_s^0(t,q^s)\left|N_s(t,q,\dot{q},\ddot{q})\right|_{\dot{q}^s=\ddot{q}^s=0}\tag{2.3}$$

then in fact $\ddot{q}^s(t) = 0$ and

$$Q_s^T(t,q,\dot{q},\ddot{q})=Q_s^{T0}(t,q,\dot{q},\ddot{q})$$

If inequality (2.3) is not satisfied, the assumption made (namely that $\ddot{q}^s = 0$) can be discarded and we can assume that

$$Q_s^T(t,q,\dot{q},\ddot{q})=f_s(t,q^s,0)|N_s(t,q,\dot{q},\ddot{q})|\operatorname{sgn}Q_s^{T0}(t,q,\dot{q},\ddot{q})\tag{2.4}$$

Then, in the actual motion of the system $\ddot{q}^s \neq 0$, since when $\ddot{q}^s = 0$ we would have obtained

$$\left|Q_s^{T0}\right|=f_s^0\left|N_s\right|_{\dot{q}^s=\ddot{q}^s=0}.$$

In the general case we obtain the following expression for the generalized force of sliding friction [9]

$$Q_s^T(t,q,\dot{q},\ddot{q})=\begin{cases}-f_s(t,q^s,\dot{q}^s)\left|N_s(t,q,\dot{q},\ddot{q})\right|\operatorname{sgn}\dot{q}^s, & \text{if}\quad \dot{q}^s\neq 0\\ f_s(t,q^s,0)\left|N_s(t,q,\dot{q},\ddot{q})\right|\operatorname{sgn}Q_s^{T0}(t,q,\dot{q},\ddot{q}), & \text{if}\quad \dot{q}^s=0\\ \qquad |Q_s^{T0}(t,q,\dot{q},\ddot{q})|>f_s^0(t,q^s)\left|N_s(t,q,\dot{q},\ddot{q})\right|_{\ddot{q}^s=0} & \\ Q_s^{T0}(t,q,\dot{q},\ddot{q}), \quad \text{if}\quad \dot{q}^s=0, & \\ \qquad |Q_s^{T0}(t,q,\dot{q},\ddot{q})|\leqslant f_s^0(t,q^s)\left|N_s(t,q,\dot{q},\ddot{q})\right|_{\ddot{q}^s=0} & \end{cases}\tag{2.5}$$

Certain constraints on the region of acceptable values of $\ddot{q}$ in R^k of the type of dependences of the values of $\ddot{q}$ on the values of t, q and $\dot{q}$ follow immediately from the equations of motion (1.3) of a system with sliding friction (2.5).

A unique supplementation of the right-hand sides of the system of equations (1.3) can nevertheless lead to a non-unique supplement of the right-hand sides of the equations of motion in the form

$$\ddot{q}=G(t,q,\dot{q})$$

i.e. after solving (1.3) for $\ddot{q}$.

3. CONVERSIONS OF THE EQUATIONS OF MOTION

The equations of motion (1.3) with the supplement (2.5) can be reduced to a form without using the conditions $\ddot{q}^s = 0$ to supplement the right-hand sides. We will put

$$N(\dot q)\overset{\Delta}{=}\left\{s\in(1,\ldots,k_*):\dot q^s=0\right\}$$

$$N_0(t,q,\dot q,\ddot q)\overset{\Delta}{=}\left\{s\in N(\dot q):|Q_s^{T0}(t,q,\dot q,\ddot q)|\leq f_s^0(t,q^s)|N_s(t,q,\dot q,\ddot q)|\right\}$$

In the general theory of mechanical systems with friction Painlevé made the assumption that forces of friction F_μ^T can be expressed as functions of t, q, $\dot q$ and F_μ^N, where F_μ^N "are in turn functions of t, q, $\dot q$ and F_μ^A, which are independent of the law of motion" (when $\dot q^i\neq 0$, $i=1,\ldots,k^*$) [1, p. 24]. If we understand this to mean that it is possible to convert the equations of motion (1.3) and (2.5) to a form in which N_s does not depend on the generalized accelerations, the conditions $\partial|N_s|/\partial\ddot q^s=0$ will be satisfied for $s=1,\ldots,k_*$.

Consider the somewhat less rigid assumption

$$f_s^0(t,q)|\partial|N_s(t,q,\dot q,\ddot q)|/\partial\ddot q^s|<a_{ss}(t,q) \tag{3.1}$$

for all $s\in N(\dot q)\backslash N_0(t,q,\dot q,\ddot q)\in\Omega\times R^k$: $|N_s(t,q,\dot q,\ddot q)|\neq 0$.

Taking into account the fact that (see [10]) the diagonal elements of the symmetrical positive definite matrix $A(t,q)$ are positive numbers, we can conclude that inequalities (3.1) are satisfied if $|N_s(t,q,\dot q,\ddot q)|$ are independent of $\ddot q^s$.

Lemma 1. If condition (3.1) is satisfied, the system of equations of motion (1.3) with the supplement (2.5) is equivalent (i.e. their solutions are, in a certain sense, identical) to the system of equations

$$\begin{aligned}
&\sum_{i=1}^k a_{si}(t,q)\ddot q^i=Q_s^{T0}(t,q,\dot q,\ddot q)+g_s(t,q,\dot q)+Q_s^A(t,q,\dot q),\quad s\in N_0(t,q,\dot q,\ddot q)\\
&\sum_{i=1}^k a_{si}(t,q)\ddot q^i=f_s(t,q^s,0)|N_s(t,q,\dot q,\ddot q)|\operatorname{sgn}Q_s^{T0}(t,q,\dot q,\ddot q)+\\
&+g_s(t,q,\dot q)+Q_s^A(t,q,\dot q),\quad s\in N(\dot q)\setminus N_0(t,q,\dot q,\ddot q)\\
&\sum_{i=1}^k a_{si}(t,q)\ddot q^i=-f_s(t,q^s,\dot q^s)|N_s(t,q,\dot q,\ddot q)|\operatorname{sgn}\dot q^s+g_s(t,q,\dot q)+\\
&+Q_s^A(t,q,\dot q),\quad s\in(1,\ldots,k_*)\setminus N(\dot q)\\
&\sum_{i=1}^k a_{si}(t,q)\ddot q^i=g_s(t,q,\dot q)+Q_s^A(t,q,\dot q),\quad s=k_*+1,\ldots,k
\end{aligned} \tag{3.2}$$

Proof. In order to prove that the system of equations (1.3), supplemented by equations (2.5), and the system of equations (3.2) are equivalent, it is sufficient to show that at each point $(t,q,\dot q)\in\Omega$ they define one and the same set of vectors $\ddot q_*$.

Suppose $\ddot q_*$ satisfies Eqs (3.2) for fixed $(t,q,\dot q)\in\Omega$. We will put

$$F_s(\ddot q^s)\overset{\Delta}{=}\frac{1}{a_{ss}(t,q)}\left[|Q_s^{T0}(t,q,\dot q,\ddot q)|-f_s^0(t,q)|N_s(t,q,\dot q,\ddot q)|\right]$$

with $\ddot q^i=\ddot q_*^i$ when $i\neq s$. If $s\in N(\dot q)\backslash N_0(t,q,\dot q,\ddot q_*)$ then $F_s(\ddot q_*^s)=\ddot q_*^s\operatorname{sgn}Q_s^{T0}(t,q,\dot q,\ddot q_*)$ and

$$|Q_s^{T0}(t,q,\dot q,\ddot q_*)|>f_s^0(t,q)|N_s(t,q,\dot q,\ddot q_*)| \tag{3.3}$$

Then $F_s(\ddot q_*^s)=|\ddot q_*^s|$. By (3.1) when $|N_s(t,q,\dot q,\ddot q)|\neq 0$ the function $F_s(\ddot q^s)$ locally satisfies the Lipschitz condition with constant $L<1$. Taking the continuity of F_s into account and also the fact that when $|N_s|=0$ the equation $F_s(\ddot q^s)=\text{const}$ holds, it can be shown that F_s satisfies the Lipschitz condition in the section $[0,\ddot q^s]$ and with constant $L<1$. Then

$$|F_s(\ddot{q}_*) - F_s(0)| \leq L|\ddot{q}_*^s - 0| < |F_s(\ddot{q}_c^s)| \tag{3.4}$$

whence it follows that $F_s(0) > 0$. This means that

$$|Q_s^{T0}(t,q,\dot{q},\ddot{q})| > f_s^0(t,q)|N_s(t,q,\dot{q},\ddot{q})|_{\ddot{q}^s=0} \tag{3.5}$$

with the condition $\ddot{q}^i = \ddot{q}_*^i$, if $i \neq s$. Hence, the right-hand sides of (3.2) and (3.1) with the supplement of generalized forces of friction (2.5) are identical at the point $(t, q, \dot{q}, \ddot{q}_*)$ for all $s \in N(\dot{q}) \backslash N_0(t, q, \dot{q}, \ddot{q}_*)$. If $s \in N_0(t, q, \dot{q}, \ddot{q}_*)$, then $\ddot{q}^s = 0$ and inequality (2.3) is satisfied for $\ddot{q} = \ddot{q}_*$. Consequently, the right-hand sides of the systems of equations (1.3), (2.5) and (3.2) are identical for all $s \in N_0(t, q, \dot{q}, \ddot{q}_*)$. For $s \in (l, \dots, k) \backslash N(\dot{q})$ Eqs (1.3) and (3.2) are obviously identical. Hence $\ddot{q}_*$ is also a solution of system of equations (1.3) and (2.5).

Conversely, suppose $\ddot{q}_*$ satisfies (1.3) and (2.5). If $s \in N(\dot{q})$ and $\ddot{q}_*^s \neq 0$, then (3.5) is satisfied. Then $F_s(0) > 0$ $|F_s(\ddot{q}_*^s)| = |\ddot{q}_*^s \operatorname{sgn} Q_s^{T0}| = |\ddot{q}_*^s|$ and (3.4) still holds. Consequently, $F_s(q_*^s) > 0$, and this denotes that inequality (3.3) holds. Hence it follows that the right-hand sides of systems (1.3), (2.5) and (3.2) are identical for all $s \in N(\dot{q})$ such that $\ddot{q}_*^s \neq 0$. If $s \in N(\dot{q})$ and $\ddot{q}_*^s = 0$, then inequality (2.3) is satisfied for $\ddot{q} = \ddot{q}_*$, and hence $s \in N_0(t, q, \dot{q}, \ddot{q}_*)$. Hence it follows that the right-hand sides of systems (1.3), (2.5) and (3.2) are identical for all $s \in N(\dot{q})$. As previously we conclude that $\ddot{q}_*$ is the solution of Eqs (3.2).

Lemma 2. Suppose $\ddot{q}_*$ is defined as the solution of the system of equations (3.2) or (1.3) and (2.5) at the point $(t, q, \dot{q}) \in \Omega$. Then, for any $s \in N(\dot{q})$ the following assertions are equivalent.

1. $\ddot{q}_*^s \neq 0$, $\operatorname{sgn} \ddot{q}_*^s = -\operatorname{sgn} Q_s^{T0}(t, q, \dot{q}, \ddot{q}_*)$.
2. Inequality (3.3) is satisfied.

Proof. Suppose $\ddot{q}_*$ satisfies system (1.3), (2.5). We will assume that Assertion 1 holds. If inequality (2.3) held for $\ddot{q} = \ddot{q}_*$, we would have $\ddot{q}_*^s = 0$, and since $\ddot{q}_*^s \neq 0$ this means that (2.3) is not satisfied. Hence

$$\ddot{q}_*^s = \frac{1}{a_{ss}(t,q)}\left[f_s(t,q^s,0)|N_s(t,q,\dot{q},\ddot{q}_*)|\operatorname{sgn}Q_s^{T0}(t,q,\dot{q},\ddot{q}_*) - Q_s^{T0}(t,q,\dot{q},\ddot{q}_*)\right] \tag{3.6}$$

Hence, taking into account the fact that $a_{ss}(t, q) > 0$, $\operatorname{sgn} \ddot{q}_*^s = -\operatorname{sgn} Q_s^{T0}(t, q, \dot{q}, \ddot{q}_*)$ we obtain (3.3). We have thereby established that Assertion 2 follows from Assertion 1.

Suppose Assertion 2 is not satisfied. Then for $\ddot{q} = \ddot{q}_*$ inequality (2.3) is not satisfied since, if we assumed it to be satisfied we would have that $\ddot{q}^s = 0$ and this would contradict (3.3). Consequently, (2.4) holds (at the point $(t, q, \dot{q}, \ddot{q}_*)$ and $\ddot{q}_*^s \neq 0$. Hence, equality (3.6) holds, whence taking (3.3) into account, we obtain $\operatorname{sgn} \ddot{q}_*^s = -\operatorname{sgn} Q_s^{T0}(t, q, \dot{q}, \ddot{q}_*)$. Assertion 1 therefore follows from Assertion 2.

If $\ddot{q}_*$ satisfies the system of equations (1.3), (2.5) and condition (3.1) is satisfied, then, as follows from Lemmas 1 and 2, the inequality

$$|Q_s^{T0}(t,q,\dot{q},\ddot{q}_*)| > f_s^0(t,q)|N_s(t,q,\dot{q},\ddot{q}_*^1,\dots,\ddot{q}_*^{s-1},0,\ddot{q}_*^{s+1},\dots,\ddot{q}_*^k)|$$

is equivalent to the conditions

$$\ddot{q}_*^s \neq 0, \quad \operatorname{sgn}\ddot{q}_*^s = -\operatorname{sgn}Q_s^{T0}(t,q,\dot{q},\ddot{q}_*), \quad (s \in N(\dot{q}))$$

Lemma 1 shows the importance of Eqs (3.2). We will call them the equations of the dynamics of mechanical system with sliding friction.

4. THE EXISTENCE OF CLASSICAL SOLUTIONS

We will introduce the following definition: a point $(t, q, \dot{q}, \ddot{q}) \in \Omega \times R^k$ is said to be singular if a number $s \in (1, \dots, k_*)$ exists such that $\dot{q}^s = 0$ and

$$|Q_s^{T0}(t,q,\dot{q},\ddot{q})| = f_s(t,q^s,0)|N_s(t,q,\dot{q},\ddot{q})|$$

Suppose $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ is the solution of system (1.3), on the right-hand side of which there are no friction forces. We will assume that there are no singular points in the neighbourhood of this point for any sufficiently small f_s, $s = 1, \ldots, k_*$. Then, Eqs (3.2) retain their structure for points $(t, q, \dot{q}, \ddot{q})$ from the neighbourhood of $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ for small f_s. If, in this case $|N_s(t, q, \dot{q}, \ddot{q})| \neq 0$, then, by using the theorem on implicit functions, we can show that the system of equations (3.2) is solvable for $\ddot{q}$ for certain small values of f_s.

We shall need some new notation in order to investigate Eqs (3.2) further.

Suppose $\beta \subset \{1, \ldots, k_*\}$ is a certain set of indices. We will denote the principal submatrix of matrix A by $A(\beta)$ (see [10, p. 30]), i.e. the matrix obtained from A are discarding from it the rows and columns with numbers belonging to β.

For each $s = 1, \ldots, k$ we put

$$\Delta_s = \begin{cases} -\operatorname{sgn} Q_s^{T0}(t,q,\dot{q},\ddot{q}), & s \in N \setminus N_0 \\ \operatorname{sgn} \dot{q}^s, & s \in (1,\ldots,k_*) \setminus N \\ 0, & s \in (k_*+1,\ldots,k) \cup N_0 \end{cases}$$

Suppose $\alpha = (\alpha_1, \ldots, \alpha_m) \subset (1, \ldots, k_*)$ is a multi-index. We will use the following notation: $f_\alpha \triangleq f_{\alpha_1} \ldots f_{\alpha_m}$, $\Delta_\alpha \triangleq \Delta_{\alpha_1} \ldots \Delta_{\alpha_m}$, $a_\alpha \triangleq \{a_{\alpha_1} \ldots a_{\alpha_m}\}$, where a_{α_θ} are the rows of the matrix A, $\partial|N_\alpha|/\partial\ddot{q} \triangleq \{\partial|N_{\alpha_1}|/\partial\ddot{q}, \ldots, \partial|N_{\alpha_m}|/\partial\ddot{q}\}$ where $\partial|N_{\alpha_\nu}|/\partial\ddot{q}$ are the gradients of the function $|N_{\alpha_\nu}|$ with respect to the vector $\ddot{q}$, and $[\sigma^{a_\alpha}_{\partial|N_\alpha|/\partial\ddot{q}} A]$ is the matrix obtained from A after replacing the rows a_α by $\partial|N_\alpha|/\partial\ddot{q}$ in it.

We can now establish the formula

$$\det\left[a_{s\nu}(t,q) - f_s(t,q^s,\dot{q}^s)\frac{\partial|N_s|}{\partial\ddot{q}^\nu}\Delta_s\right]_1^k (N_0) = \det A(t,q)(N_0) + \tag{4.1}$$
$$+\sum_{m=1}^{k_*}\sum_{\alpha\in C^m_{k_*}} \det\left[\sigma^{a_\alpha}_{\partial|N_\alpha|/\partial\ddot{q}} A(t,q)\right](N_0) f_\alpha \delta_\alpha$$

where $C^m_{k_*}$ is the set of all combinations of $\{1, \ldots, k_*\}$ elements taken m at a time. Since the principal submatrix of a positive-definite matrix is itself positive-definite [10, p. 472], the determinant on the left-hand side of (4.1) will be non-zero if

$$\sum_{m=1}^{k_*}\sum_{\alpha\in C^m_{k_*}} \left|\det\left[\sigma^{a_\alpha}_{\partial|N_\alpha|/\partial\ddot{q}} A(t,q)\right](N_0)\right| f_\alpha |\delta_\alpha| < \det A(t,q)(N_0) \tag{4.2}$$

Inequality (4.2) will always be satisfied for sufficiently small f_s.

We will assume that the functions Q_s^A, Q_s^*, f_s, $|N_s|$ are continuous over the set of their arguments, and the matrix $F(t, q)$ and the functions a_{si} and g_i satisfy the assumptions in Section 1.

Theorem. Suppose that at a certain non-singular point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$: $|N_s(t_0, q_0, \dot{q}_0, \ddot{q}_0)| \neq 0$, Eqs (3.2) are solvable for q_0 and the inequality (4.2) is satisfied in it for the sets of indices $N(\dot{q}_0)$, $N_0(t_0, q_0, \dot{q}_0, \ddot{q}_0)$. A classical solution $q(t)$ of the Cauchy problem (3.2) then exists, $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$, defined in a certain interval $[t_0, t_0 + \delta)$ (when $t = t_0$ the right derivative $D^+\dot{q}(t)$ satisfies Eq (3.2)). If condition (3.1) is then satisfied, a local classical solution (which emerges from this point) exists of the equations of motion (1.3) of the mechanical system with sliding friction (2.5).

Proof. We choose a neighbourhood $S_0 \subset \Omega \times R^k$ of the point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ such that for all $(t, q, \dot{q}, \ddot{q}) \in S_0$ the following conditions are satisfied.

$\dot{q}^s$ preserve the sign $\dot{q}_0^s \neq 0$ for any $s \in (1, \ldots, k_*)\backslash N(\dot{q}_0)$, $Q_s^{T0}(t, q, \dot{q}, \ddot{q})$ and $Q_s^{T0}(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ $s \in N(\dot{q}_0)\backslash(t_0, q_0, \dot{q}_0, \ddot{q}_0)$, preserve the sign for $s \in (\dot{q}_0)\backslash N(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ and $Q_s^{T0}(t, q, \dot{q}, \ddot{q})| < f_s^0(t, q^s)|N_s(t_0, q_0, \dot{q}_0, \ddot{q}_0)|$ preserve the sign for $s \in N_0(t_0, q_0, \dot{q}_0, \ddot{q}_0)$.

$s \in N(\dot{q}_0)\backslash N(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ preserve the sign for Q_s^{T0} $(t, q, \dot{q}, \ddot{q})$.

The structure of the right-hand side of (3.2) generated by the point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ is fixed on S_0. This is possible by virtue of the assumptions that the functions considered are continuous and that the point $(t_0, q_0, \dot{q}_0, \ddot{q}_0)$ is non-singular.

In the first group of equations (3.2) (i.e. for $s \in N_0$) we obtain $\ddot{q}^s = 0$. We will consider the remaining equations of (3.2) as a system of functional equations which define the implicit functions

$$\ddot{q}^s = G^s(t, q, \dot{q}), \quad s \in (1, \ldots, k) \setminus N_0 \tag{4.3}$$

By the theorem of implicit functions, by virtue of (4.2) and other assumptions, the functions G^s in (4.3) exist, are positive-definite and are continuous in a certain neighbourhood $\Omega_0 \subset \Omega$ of the point $(t_0, q_0, \dot{q}_0)$.

We will consider the system of equations (4.3) by adding to it the equations $\ddot{q}^s = G^s(t, q, \dot{q}) \equiv 0$, $s \in N_0$ with initial conditions $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$. By Peano's theorem a classical solution $q(t)$ of this Cauchy problem exists, defined in a certain interval $(t_0 - \tau, t_0 + \delta)$.

It remains to check that the function $q(t)$ satisfies the system of equations (3.2) in the interval $[t_0, t_0 + \delta]$. This is known for the first group of equations and it is obvious for the third and fourth groups of equations (3.2).

For the second group, by virtue of Lemma 2, $\operatorname{sgn} \ddot{q}^s(t_0) = \operatorname{sgn} Q_s^{T0}(t_0, q_0, \dot{q}_0, \ddot{q}_0)$, $s \in N \backslash N_0$. Consequently, $\dot{q}^s(t) \neq 0$, $\operatorname{sgn} \dot{q}^s(t) = -\operatorname{sgn} Q_s^{T0}(t_0, q_0, \dot{q}_0, \ddot{q}_0)$, for $t \in [t_0, t_0 + \delta]$, $s \in N \backslash N_0$. The second group of equations (3.2), which define the implicit function G, along the solution $q(t)$, reduces the third group of equations (3.2) when $t \in (t_0, t_0 + \delta)$. Hence, it has been proved that $q(t)$ is a solution of the system of equations (3.1) with second derivative $\ddot{q}(t)$ continuous in $[t_0, t_0 + \delta]$, and since $\ddot{q}(t_0) = G(t_0, q_0, \dot{q}_0) = D^+\dot{q}(t_0)$, we have that $D^+\dot{q}(t_0)$ also satisfies (3.1).

Hence, we have proved the existence of a solution of system of equations (3.2). If condition (3.1) is satisfied, then, by Lemma 1, this solution will simultaneously be a solution of system of equations (1.3) and (2.5). This proves the theorem.

Note. An advantage of the equations of dynamics (3.2) is the simpler conditions for defining their right-hand sides (compared with the supplement (2.5) of the equations of motion (1.3) by the rules of classical mechanics), which turn out to be continuous with respect to $\ddot{q} \in R^k$ for any fixed $(t, q, \dot{q}) \in \Omega$. In addition, Lemma 2 shows that the vector of generalized accelerations $\ddot{q}_*$ considered as an implicit function of $(t, q, \dot{q})$, defined from (3.2), possesses the following important property

$$s \in N(\dot{q}), \ \ddot{q}_*^s \neq 0 \Rightarrow \operatorname{sgn} \ddot{q}_*^s = -\operatorname{sgn} Q_s^{T0}(t, q, \dot{q}, \ddot{q}_*)$$

which ensures that $\ddot{q}_*$ is continuous with respect to $(t, q, \dot{q})$ along specially chosen sets. Hence (bearing Lemma 1 in mind) we will henceforth consider Eqs (3.2) of the dynamics of mechanical systems with sliding friction.

5. PAINLEVÉ'S EXAMPLE ([1], SEE ALSO [3])

The heavy material points of unit mass connected by a weightless rod of length $r > 0$ are considered. One of them slides with friction along a fixed horizontal straight line Ox (its coordinate is x and the reaction of the axis is (N_1, F_1^T)), and the other moves without external resistance in a vertical plane Oxy under the action of gravity g (and the reaction of the rod). The Oy axis is directed downwards and θ is the angle of inclination of the rod in a clockwise direction from the positive direction of Ox. The equations of motion of the system can be written in Lagrangian form as

$$\begin{aligned} 2\ddot{x} - r\sin\theta\ddot{\theta} &= r\dot{\theta}^2\cos\theta + Q_1^T \\ -r\sin\theta\ddot{x} + r^2\ddot{\theta} &= rg\cos\theta \end{aligned} \tag{5.1}$$

The generalized forces of the normal reaction and static friction are as follows:

$$|N_1(\theta, \dot{\theta}, \ddot{\theta})| = |r(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) - 2g|$$

$$Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta}) = -r\sin\theta\ddot{\theta} - r\dot{\theta}^2\cos\theta$$

By Coulomb's laws for the force of dry dynamic friction when $\dot{x} \neq 0$

$$Q_1^T(\theta, \dot{x}, \dot{\theta}, \ddot{\theta}) = -f|N_1(\theta, \dot{\theta}, \ddot{\theta})| sgn\dot{x} \tag{5.2}$$

where $f > 0$ is the coefficient of dynamic friction. The equations of motion reduce to the form

$$\left\| \begin{matrix} \ddot{x} \\ \ddot{\theta} \end{matrix} \right\| = A^{-1}(\theta) \left\| \begin{matrix} r\dot{\theta}^2 \cos\theta - f|r(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) - 2g| \operatorname{sgn}\dot{x} \\ rg\cos\theta \end{matrix} \right\| \tag{5.3}$$

For sufficiently small $f > 0$ these equations are solvable for $\ddot{x}$, $\ddot{\theta}$

$$\ddot{x} = G_1(\theta, \dot{x}, \dot{\theta}, f), \quad \ddot{\theta} = G_2(\theta, \dot{x}, \dot{\theta}, f) \tag{5.4}$$

and the right-hand sides G_1 and G_2 are continuous with respect to $\theta, \dot{x}, \dot{\theta}, f$. For any $t_0, x_0, \theta_0, \dot{x} \neq 0, \dot{\theta}_0$, local classical solution of Eqs (5.4) exists, and so also of Eqs (5.1) and (5.2). The sufficient condition for Eqs (5.3) to be solvable for $\ddot{x}$, $\ddot{\theta}$ is

$$f < (1 + \cos^2\theta) / |\sin\theta\cos\theta| \tag{5.5}$$

Suppose now that, at certain instants of time $t > t_0$, the equality $\dot{x}(t) = 0$ can be satisfied by Coulomb's law the conditions of equilibrium of a point on the straight line Ox will be

$$\ddot{\theta} = gr^{-1}\cos\theta, \quad \ddot{x} = 0, \quad |Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta})| \leqslant f|N_1(\theta, \dot{\theta}, \ddot{\theta})|$$

If $|Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta})| > f|N_1(\theta, \dot{\theta}, \ddot{\theta})|$, the conditions of equilibrium break down and $Q_1^{T0} = -f|N_1| \operatorname{sgn} \ddot{x}$. Taking into account the fact that when $\dot{x}(t) = 0$ we will have $\operatorname{sgn}\ddot{x} = -\operatorname{sgn} Q_1^{T0}$, we obtain

$$Q_1^T(\theta, \dot{x}, \dot{\theta}, \ddot{\theta}) = \begin{cases} Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta}), & \text{if} \quad \dot{x} = 0 \text{ and} \\ & |Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta})| \leqslant f|N_1(\theta, \dot{\theta}, \ddot{\theta})| \\ -f|N_1(\theta, \dot{\theta}, \ddot{\theta})| \operatorname{sgn} Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta}), & \text{if} \quad \ddot{x} = 0 \text{ and} \\ & |Q_1^{T0}(\theta, \dot{\theta}, \ddot{\theta})| > f|N_1(\theta, \dot{\theta}, \ddot{\theta})| \\ -f|N_1(\theta, \dot{\theta}, \ddot{\theta})| \operatorname{sgn}\ddot{x}, & \text{if} \quad \dot{x} \neq 0 \end{cases} \tag{5.6}$$

The sets

$$N = \begin{cases} \{1\}, & \text{if} \quad \dot{x} = 0 \\ \emptyset, & \text{if} \quad \dot{x} \neq 0 \end{cases}$$

$$N_0 = \begin{cases} \{1\}, & \text{if} \quad \dot{x} = 0, \ |Q_1^{T0}| \leqslant f|N_1| \\ \emptyset, & \text{if} \quad (\dot{x} \neq 0) \vee (\dot{x} = 0, |Q_1^{T0}| > f|N_1|) \end{cases}$$

These inequalities do not contain $\ddot{x}$. Hence, the equations of dynamics (of the type (3.2)) of the system considered are identical with the equations of motion (5.1) with supplement (5.6). Condition (3.1) is satisfied since $\partial|N_1|/\partial\ddot{x} = 0$.

Consider (4.2) as it applies to Eqs (5.1). Here

$$k_* = 1, m = 1, \Delta_2 = 0$$

$$\Delta_1 = \begin{cases} -\operatorname{sgn} Q_1^{T0}, & \dot{x} = 0, N_0 = \emptyset \\ \operatorname{sgn}\dot{x}, & \dot{x} \neq 0 \\ 0 & N_0 = \{1\} \end{cases}$$

$$\det A(t,q)(N_0) = \det A(\theta)(N_0) = \begin{Bmatrix} r^2(1+\cos^2\theta), & N = \varnothing \\ r^2, & N_0 = \{1\} \end{Bmatrix}$$

$$\sum_{m=1}^{k_*} \sum_{\alpha\in C_{k_*}^m} \det\left[\sigma^{a_\alpha}_{\partial|N_\alpha|/\partial\ddot{q}} A(t,q)\right](N_0)|f_\alpha|\delta_\alpha| =$$

$$=|\det\begin{Vmatrix} 0, & \partial|N_1|/\partial\ddot{\theta} \\ -r\sin\theta, & r^2 \end{Vmatrix}(N_0)|f|\Delta_1| = \begin{cases} 0, & \text{if} \quad N_0 = \{1\} \\ r^2 f|\cos\theta\sin\theta|, & \text{if} \quad N_0 = \varnothing \end{cases}$$

Formula (4.2) has the form

$$r^2 > 0, \quad \text{if} \quad N_0 = \{1\}:$$

$$1 + \cos^2\theta > f|\sin\theta\cos\theta|, \quad \text{if} \quad N_0 = \varnothing$$

The first of the inequalities obtained is obviously satisfied, while the second is identical with (5.5) and gives the sufficient condition for a local classical solution of problem (5.1) to exist for each non-singular initial point (x_0, θ_0, $\dot{x}_0$ $\dot{\theta}_0$, $\ddot{x}_0$, $\ddot{\theta}_0$), which satisfies (5.1) and such that $\ddot{\theta}_0\cos\theta_0 \neq \dot{\theta}_0^2 \sin\theta_0 + 2g/r$.

This research was carried out with support from the Russian Fund for Fundamental Research (93-013-16295).

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Translated by R.C.G.